# List of open problems

**Problem 1** (G. Traustason). *Let $G$ be a residually finite 2-group and $x \in G$ a left 3-Engel element of order 2. Is $\langle x \rangle^G$ locally nilpotent?*

**Remark**. This is true if $x$ is of odd order, even without the assumption that $G$ is residually finite. [E. Jabara and G. Traustason, Left 3 -Engel elements of odd order in groups. Proc. Amer. Math. Soc. 147 (2019), no. 5, 1921—1927.]

**Problem 2** (P. Shumyatsky). *Let $G$ be a linear group and $X$ the set of almost Engel elements. Is $X$ a subgroup of $G$? (An element $g$ in $G$ is said to be almost Engel if for every $x \in G$, there exists a finite set $\mathcal{E}_x$ and a positive integer $n(x)$ such that $[x,_{n(x)} g] \in \mathcal{E}_x$.)*

**Problem 3** (M. Noce and G. Tracey). *Let $T_d$ be a regular rooted tree and $Aut T_d$ its group of automorphisms. A subgroup $G \leq Aut T_d$ is a branch group [6] if*

*(i) $G$ is spherically transitive;*

*(ii) For all $i \geq 1$, there exists $L_i \leq Aut T_d$ such that*

$$H_i := \underbrace{L_i \times \cdots \times L_i}_{d^i} \leq stab_G(i)$$

*is a normal subgroup of finite index in $G$.*

*Can a finitely generated infiinite branch group be an Engel group?*

**Problem 4** (M. Maj and P. Longobardi). *Let $G$ be an $n$-Engel torsion-free group, $S \subseteq G$, $S$ finite.*

*(a) Suppose $|S^2| \leq 3|S| - 4$, where $S^2 = \{xy | x, y \in S\}$. Is $\langle S \rangle$ abelian?*

*(b) Is there a positive integer $b$ where $b \geq 4$ such that whenever $|S^2| \leq 2|S| - 1 + b$, then $\langle S \rangle$ is abelian?*

**Problem 5** (G. Traustason). *Let $G$ be a torsion-free group for which every subgroup is subnormal of defect at most $n$. Is $G$ nilpotent of class at most $n$?*

**Remark**. This is true for $n \leq 4$. (H. Smith and G. Traustason, Torsion-free groups with all subgroups 4-subnormal. Comm. Alg. , **33** No. 12 (2005), 4567-4585) .

**Problem 6** (C. Monetta, A. Tortora). *Let $F$ be the free group on free generators $x_1, x_2, \dots, x_m$. A group-word $w = w(x_1, \dots, x_m)$ is any nontrivial element of $F$, that is, a product of finitely many $x_i$'s and their inverses. The word*

$$[x_{i_1}, x_{i_2}, \dots, x_{i_k}] = [\dots[[x_{i_1}, x_{i_2}], x_{i_3}], \dots, x_{i_k}]$$

*is said to be a simple commutator word if $k \geq 2, i_1 \neq i_2$ and $i_j \in \{1, \dots, m\}$ for every $j \in \{1, \dots, k\}$ (see [8]). Examples of simple commutator words are the lower central words $\gamma_k = [x_1, \dots, x_k]$ and the $n$-Engel word $[x,_n y]$.*

*It is easy to see that if $G$ is a group satisfying the law*

$$w = [x_{i_1}, x_{i_2}, \dots, x_{i_k}] = 1$$

*where $w$ is a simple commutator word and $k \leq 4$, then $G$ is locally nilpotent. Is this also true for $k = 5$?*

**Problem 7** (A. Tortora, M. Tota). *The origin of the general Burnside problem is a famous paper of 1902, where Burnside asked whether a finitely generated periodic group must be finite [3]. This problem remained unsolved until 1964 when Golod gave an example of a finitely generated infinite $p$-group, which is also residually finite [4]. Later other examples have been found (see, for instance, [5] and [7]).*

*So far the group constructed by Golod is the only example of a finitely generated non-nilpotent Engel group. Find another example of a residually finite Engel group which is not locally nilpotent. (Notice that the first Grigorchuk group and the Gupta-Sidki 3-group are not Engel groups [1]).*

**Problem 8** (A. Tortora). *According to a result of Bartholdi [1] (see also [2]), the first Grigorchuk group is not an Engel group. This is an immediate consequence of the fact that involutions are the only left Engel elements of the first Grigorchuk group. The proof relies on a GAP calculation. Find a computer-free proof of this latter result.*

**Problem 9** (A. Abdollahi). *Let $G$ be a Hausdorff compact group and such that the set $E$ of values of $k$-Engel word on $G$ has positive Haar measure for some positive integer $k$.*

*a) Is it true that $G$ contains an open $k$-Engel subgroup?*
*b) Is it true that $E$ contains a non-empty open subset?*

**Problem 10** (A. Abdollahi). *Let $G$ be an infinite group such that every infinite subset of $G$ contains two distinct elements generating a $k$-Engel subgroup. Is it true that there exists a finite upper bound on the size of all (finite) subsets of $G$ all of whose pairwise distinct elements generate a non-$k$-Engel subgroup?*

**Problem 11** (A. Abdollahi). *Let $G$ be an infinite group such that every two infinite subsets $X$ and $Y$ of $G$ contain elements $x$ and $y$, respectively, such that $[x,_k y] = 1$. Is it true that $G$ is $k$-Engel? The answer is positive for $k \leq 3$.*

# References

[1] L. Bartholdi, *Algorithmic decidability of Engel's property for automaton groups*, Computer Science–Theory and Applications, 29–40, Springer, 2016.

[2] L. Bartholdi, *Decidability problems in automaton semigroups*, arXiv:1705.04598 [math.GR].

[3] W. Burnside, *On an unsettled question in the theory of discontinous groups*, Quart. J. Pure Appl. Math. **33** (1902), 230–238.

[4] E. S. Golod, *Some problems of Burnside type*, 1968 Proc. Int. Congr. Math. (Moscow 1966), 284–289; *Amer. Math. Soc. Transl.* (2) **84** (1969), 83–88.

[5] R. I. Grigorchuk, *On Burnside's problem on periodic groups*, Funktsional. Anal. i Prilozhen. **14** (1980), 53–54.

[6] R. I. Grigorchuk, *Just infinite branch groups*, New Horizons in pro $p$-groups, 2000, 121–179.

[7] N. Gupta and S. Sidki, *On the Burnside problem for periodic groups*, Math. Z. **182** (1983), no. 3, 385–388.

[8] C. Monetta and A. Tortora, *A nilpotency criterion for some verbal subgroups*, Bull. Aust. Math. Soc. **100** (2019), no. 2, 281–289.